Journal of Engineering Mathematics, Vol. 11, No. 3, July 1977
Noordhoff International Publishing–Leyden
Printed in The Netherlands

A comparison of boundary methods for the numerical solution of hyperbolic systems of equations

J. S. BRAMLEY and D. M. SLOAN

Department of Mathematics, University of Strathclyde, Glasgow, Scotland

(Received October 27, 1975)

SUMMARY

This paper compares the numerical solution of a linear system of hyperbolic partial differential equations in one and two space dimensions with the analytic solution. A two step Lax-Wendroff difference scheme is used in the interior region and various methods are used at the boundaries. The accuracy of the overall solution is tabulated for each of the boundary methods. Of particular interest here is the accuracy of the various boundary methods which are used.

1. Introduction

The motivation for this paper arose from consultation with civil engineers who were interested in mathematical modelling of estuary flows. The computational work arising in this field involves the solution of a system of nonlinear hyperbolic equations in one or two space dimensions and time. A great deal of work has been done on the analysis of pure initial value problems involving hyperbolic systems, but the analysis of initial-boundary value problems is not so well developed. Difference schemes often require more boundary conditions than the differential equations which they approximate and the inclusion of extra boundary conditions often leads to instabilities and inaccuracies. This type of problem is now attracting a great deal of theoretical interest. For example, Koster [1] has studied a constant coefficient hyperbolic system in one space dimension and he has shown that if an extra boundary condition is applied to a component of the solution corresponding to a characteristic curve directed away from the boundary then a certain type of instability will result. Roache [2] surveys methods for the numerical solution of partial differential equations in finite regions with reference to fluid mechanics.

The Fourier stability analysis which is used for initial value problems cannot, in general, be applied to initial-boundary value problems and one has to resort to energy methods or to normal mode analysis. Both methods are described in Richtmyer and Morton [3]. Kreiss [4] has extended the latter approach and, with others, he has examined various boundary conditions which might be used with some popular difference schemes.

There is a dearth of test computations on boundary techniques which might be useful to the non-specialists who find themselves involved in this type of work. Gourlay and Morris [5] have discussed some boundary adaptations of an optimally stable scheme and they have presented some test computations on a two space dimensional problem. A recent paper by Chu and Sereny [6] describes some test calculations on one dimensional isentropic

flow of a polytropic gas. Various boundary techniques were used and comparisons were made in terms of accuracy. The stability of the methods used in this study has been established by Sundstrom [7].

The present paper describes a series of test computations in one and two space dimensions. The two step L–W method is used in the interior region and, in the case of the 2 dimensional problem, the optimally stable rotated Richtmyer variety is used as described by Wilson [8]. The objective is to compare the accuracy of the overall solution using various modes of approximation at the boundaries. Boundary approximations are used which have been found to be stable, but no stability analysis is presented. For the interior region the stability analysis is found in [3]. At the boundaries the stability is not analysed, but when characteristics or their derivatives are used the stability conditions for the characteristics are not violated. This paper is not an attempt to find the best method for solving the one and two dimensional wave equations. These equations are used because they are the simplest equations with positive and negative characteristic slopes and their analytic solutions are easily derived. The approach of this paper is empirical rather than theoretical and it is intended to give the non-specialist some idea of the best boundary method to use and the accuracy to expect.

2. Boundary conditions

A linear, constant coefficient hyperbolic system in one space dimension may be written in the form

$$\frac{\partial \boldsymbol{U}}{\partial t} + A\frac{\partial \boldsymbol{U}}{\partial x} = \boldsymbol{C}, \tag{2.1}$$

where $\boldsymbol{U} = \boldsymbol{U}(x, t)$ is an n-vector and A is a constant $n \times n$ matrix. Hyperbolicity implies that A has n real eigenvalues and a complete set of linearly independent eigenvectors. There exists a non-singular matrix P such that $PAP^{-1} = D$, where D is a diagonal matrix having eigenvalues of A as elements. Suppose that the first r diagonal elements of D are non-positive and the remaining $n-r$ are positive. The transformation $\boldsymbol{V} = P\boldsymbol{U}$ enables equation (2.1) to be written as n decoupled scalar equations

$$\frac{\partial V_j}{\partial t} + \lambda_j \frac{\partial V_j}{\partial x} = [P\boldsymbol{C}]_j, \tag{2.2}$$

where $\lambda_j \leqq 0$ for $j = 1, 2, \ldots, r$ and $\lambda_j > 0$ for $j = r+1, r+2, \ldots, n$.

Suppose now that the given system is being solved between the boundaries $x = 0$ and $x = 1$. On the left hand boundary the quantities V_j, $j = 1, 2, \ldots, r$ may be obtained by integrating along the characteristics $dx/dt = \lambda_j$ which are directed towards that boundary from the interior region. The remaining $n-r$ quantities have to be specified on $x = 0$, or given in terms of V_j, $j = 1, 2, \ldots, r$. Thus, on $x = 0$, the analytic boundary conditions will have the form

$$\boldsymbol{V}^{(2)}(0, t) = R_0 \boldsymbol{V}^{(1)}(0, t) + \boldsymbol{g}_0(t), \tag{2.3}$$

where $\boldsymbol{V}^{(1)}$ represents the first r components of $\boldsymbol{V}$ and $\boldsymbol{V}^{(2)}$ represents the remaining $n-r$ components. R_0 is an $(n-r) \times r$ matrix and $\boldsymbol{g}_0(t)$ is an $(n-r)$ vector. In summary, on the

boundary $x = 0$ towards which r characteristics are directed, r quantities may be calculated using these characteristics and $n - r$ boundary conditions have to be supplied. The latter must be in a form which enables the remaining $n - r$ elements of V to be calculated. An analogous situation holds on the boundary at $x = 1$.

If $A = A(x, t)$ the situation is very similar to that described above. The essential boundary conditions at a boundary point are determined by the orientation of the characteristics at that point. An ideal numerical method for dealing with boundaries will in some sense approximate the analytic boundary situation.

A linear hyperbolic system in two space dimensions may be written as

$$\frac{\partial \boldsymbol{U}}{\partial t} + A\frac{\partial \boldsymbol{U}}{\partial x} + B\frac{\partial \boldsymbol{U}}{\partial y} = \boldsymbol{C} \tag{2.4}$$

On a boundary $x =$ constant, $n - r$ conditions are needed if A has r negative eigenvalues and the system is being solved in the region which lies on the positive side of this boundary. As in the one dimensional case, a "good" approximate method will make use of this information.

3. Description of test problems

For the one dimensional test problem we chose to solve the constant coefficient wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \tag{3.1}$$

in the region $0 < x < 1$, $t > 0$, with $u(x, t)$ subject to the boundary conditions

$$u(0, t) = u(1, t) = 0 \text{ for } t > 0,$$

and the initial conditions

$$u = 0.05 \cos 2\pi x, \quad \frac{\partial u}{\partial t} = 0 \text{ for } t = 0 \text{ and } 0 < x < 1.$$

Detailed difference equations for this case are not given but may be derived from the two dimensional case described below on neglecting the y coordinate.

For the two dimensional test problem we considered the solution of

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{3.2}$$

in $-1 < x < 1$, $0 < y < 1$, $t > 0$, with $u(x, y, t)$ subject to the initial conditions $u = \sin \pi x \sin \pi y$, $\partial u/\partial t = 0$ at $t = 0$ throughout the domain of the problem and with $u = 0$ on the boundary of this domain for all time. The symmetry of the problem in the x direction is used and the solution of equation (3.2) is computed in the region $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. The analytic solution of this latter problem is

$$u(x, y, t) = \sin \pi x \sin \pi y \cos \sqrt{2}\, c\pi t. \tag{3.3}$$

Prior to considering the numerical solution of (3.2) we write the equation in the form

$$\frac{\partial \boldsymbol{U}}{\partial t} + A\frac{\partial \boldsymbol{U}}{\partial x} + B\frac{\partial \boldsymbol{U}}{\partial y} = \boldsymbol{0}, \tag{3.4}$$

where

$$\boldsymbol{U} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial t} \\ c\dfrac{\partial u}{\partial x} \\ c\dfrac{\partial u}{\partial y} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -c & 0 \\ -c & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & -c \\ 0 & 0 & 0 \\ -c & 0 & 0 \end{bmatrix}.$$

When expressed in conservation form equation (3.4) becomes

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{U}) + \frac{\partial}{\partial y}\boldsymbol{G}(\boldsymbol{U}) = \boldsymbol{0} \tag{3.5}$$

where $\boldsymbol{F}^T = [-cU_2, -cU_1, 0]$ and $\boldsymbol{G}^T = [-cU_3, 0, -cU_1]$.

The initial conditions are

$$\left.\begin{aligned} &U_1(x, y, 0) = 0, \\ &U_2(x, y, 0) = c\pi \cos \pi x \sin \pi y, \\ \text{and } &U_3(x, y, 0) = c\pi \sin \pi x \cos \pi y. \end{aligned}\right\} \tag{3.6}$$

The boundary conditions on $\boldsymbol{U}$ may be written as

$$U_1 = U_3 = 0 \text{ on } x = 1$$

and

$$U_1 = U_2 = 0 \text{ on } y = 0 \text{ and } y = 1. \tag{3.7}$$

Also, U_2 is an even function of x whilst U_1 and U_3 are odd functions of x.

4. Numerical solution

The numerical solution is calculated on a mesh $(j\Delta x, k\Delta y)$ where j, k are integers and $0 \leqq j \leqq 20$, $0 \leqq k \leqq 20$. The solution at the interior points is obtained by considering the conservation equation (3.5) and using a 2 step Lax-Wendroff difference scheme accurate to second order. The particular variation used is the rotated Richtmyer scheme described by Wilson [8]. The two solution steps may be written as

$$\boldsymbol{U}^{n+\frac{1}{2}}_{j+\frac{1}{2},k+\frac{1}{2}} = \tfrac{1}{4}(\boldsymbol{U}^n_{j,k} + \boldsymbol{U}^n_{j+1,k} + \boldsymbol{U}^n_{j,k+1} + \boldsymbol{U}^n_{j+1,k+1}) - \frac{\Delta t}{4\Delta x}(\boldsymbol{F}^n_{j+1,k+1} + \boldsymbol{F}^n_{j+1,k}$$

$$- \boldsymbol{F}^n_{j,k+1} - \boldsymbol{F}^n_{j,k}) - \frac{\Delta t}{4\Delta y}(\boldsymbol{G}^n_{j,k+1} + \boldsymbol{G}^n_{j+1,k+1} - \boldsymbol{G}^n_{j,k} - \boldsymbol{G}^n_{j+1,k}), \tag{4.1}$$

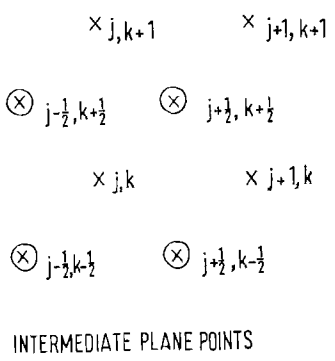


Figure 1. Mesh points for interior difference scheme.

$$U_{j,k}^{n+1} = U_{j,k}^{n} - \frac{\Delta t}{2\Delta x}(F_{j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} + F_{j+\frac{1}{2},k-\frac{1}{2}}^{n+\frac{1}{2}} - F_{j-\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - F_{j-\frac{1}{2},k-\frac{1}{2}}^{n+\frac{1}{2}})$$

$$- \frac{\Delta t}{2\Delta y}(G_{j-\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} + G_{j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - G_{j+\frac{1}{2},k-\frac{1}{2}}^{n+\frac{1}{2}} - G_{j-\frac{1}{2},k-\frac{1}{2}}^{n+\frac{1}{2}}), \tag{4.2}$$

where $U_{j,k}^{n}$ denotes the approximation to U at time t_n at point (j, k) and where the mesh points are as shown in Figure 1. If $\Delta x = \Delta y$ then the stability condition for this optimally stable scheme is the Courant-Friedrichs-Lewy (CFL) condition $c(\Delta t/\Delta x) \leqq 1$. Intermediate solution points on the plane $t = t_{n+\frac{1}{2}} = t_n + \frac{1}{2}\Delta t$ are calculated at $(-\frac{1}{2}\Delta x, k\Delta y)$ using the symmetry of the problem so that the mesh points along the y axis can be treated as interior mesh points. At the physical boundaries the various methods used are summarised below and labelled for future reference.

A) *Analytic boundary method*

The analytic solution is substituted at the boundary and the errors arising from the interior numerical approximation are monitored.

B) *Zero order extrapolation boundary method*

The value of a dependent variable at a boundary node is set equal to the value of that variable at the nearest interior node on the same time step.

C) *Characteristic boundary method*

(i) One dimension.
The characteristics in the one dimensional problem are $dx/dt = \pm c$ and the compatibility conditions are

$$\frac{dU_1}{dt} - \frac{dU_2}{dt} = 0 \text{ on } \frac{dx}{dt} = c \tag{4.2}$$

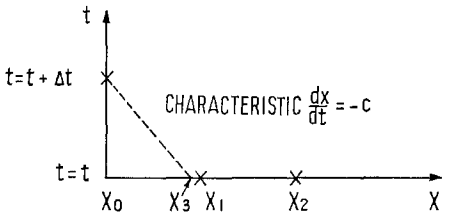


Figure 2. Characteristic for 1D at the boundary.

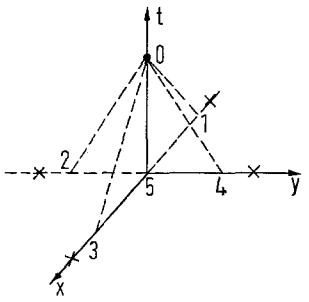


Figure 3. Characteristics for 2D.

and

$$\frac{dU_1}{dt} + \frac{dU_2}{dt} = 0 \text{ on } \frac{dx}{dt} = -c, \tag{4.3}$$

where d/dt denotes a total time derivative. On the boundary $x = 0$ the first component of $\boldsymbol{U}$ is zero since $u(0, t) = 0$ and a method is required which will enable U_2 to be computed. As suggested in Section 2, the boundary solution is obtained using the characteristics $dx/dt = -c$ which are directed towards the boundary from the region $x > 0$. Thus, for the left hand boundary we use

$$U_1(x_0, t + \Delta t) - U_1(x_3, t) + U_2(x_0, t + \Delta t) - U_2(x_3, t) = 0,$$

which is a difference approximation to equation (4.3) in the notation of Figure 2. Since $U_1(x_0, t + \Delta t) = 0$ this becomes

$$U_2(x_0, t + \Delta t) = U_2(x_3, t) + U_1(x_3, t). \tag{4.4}$$

The values of U_1 and U_2 at the point x_3 are calculated using quadratic interpolation over the points x_0, x_1 and x_2. The condition $c(\Delta t/\Delta x) \leqq 1$, which is necessary for the stability of the L–W solution in the interior region, will ensure that the negative characteristic through $(x_0, t + \Delta t)$ will meet time t between x_0 and x_1.

(ii) Two dimensions.

The characteristics boundary method used in the two dimensional problem is that described by Butler [9]. In Figure 3 the point 0 is at time $t + \Delta t$ and the set of points 1, 2, 3, 4 and 5 is at time t: approximations to variables at a point in the set may be obtained using quadratic interpolation over nodes at time t. Note that time t here coincides with time t_n

referred to in equations (4.1) and (4.2). The single infinity of bicharacteristic directions through the point 0 is expressed by

$$dx = c\cos\theta\, dt, \quad dy = c\sin\theta\, dt, \tag{4.5}$$

where $0 \leqq \theta < 2\pi$. The compatibility condition along a bicharacteristic given in Butler [9] is

$$\frac{dU_1}{dt} - \cos\theta\frac{dU_2}{dt} - \sin\theta\frac{dU_3}{dt} = c\sin^2\theta\frac{\partial U_2}{\partial x} - c\sin\theta\cos\theta\left(\frac{\partial U_2}{\partial y} + \frac{\partial U_3}{\partial x}\right) + c\cos^2\theta\frac{\partial U_3}{\partial y}, \tag{4.6}$$

where d/dt denotes total differentiation along the direction given by (4.5). The bicharacteristic θ through the point 0 will meet the plane at time t at one of the points 1, 2, 3 or 4 if θ is selected appropriately. For example, $\theta = 0$ corresponds to a bicharacteristic through the point 1 and the segment of the bicharacteristic between times t and $t + \Delta t$ has top point 0 and bottom point 1. To approximate equation (4.6), a total derivative dU_i/dt ($i = 1, 2, 3$) is replaced by $(U_i(\text{top}) - U_i(\text{bottom}))/\Delta t$ and the right hand side of the equation is replaced by the mean of the values at the top and bottom of the bicharacteristic.

In addition to the characteristic cone defined by (4.5) there is the degenerate cone $dx = dy = 0$ on which we have the compatibility relation $dU_1/dt = c(\partial U_2/\partial x + \partial U_3/\partial y)$. This characteristic is not required, however, in solving at boundary points in the chosen problem.

Referring to Figure 3, the boundary will pass through the points 0,5 and either 1 and 3 or 2 and 4. The unknown dependent variable at the boundary is calculated using the bicharacteristic with top at 0, bottom in the solution region, and whose projection on the plane t = constant is perpendicular to the boundary. Along the boundary $x = 1$ the appropriate bicharacteristic is given by $\theta = 0$ and the compatibility condition becomes

$$\frac{dU_1}{dt} - \frac{dU_2}{dt} = c\frac{\partial U_3}{\partial y}. \tag{4.7}$$

At the point j in Figure 3 the value of U_i is denoted by $(U_i)_j$ so the difference approximation to equation (4.7) becomes

$$(U_1)_0 - (U_1)_1 - (U_2)_0 + (U_2)_1 = c\frac{\Delta t}{2}[(U_{3,y})_0 + (U_{3,y})_1], \tag{4.8}$$

where $U_{3,y}$ denotes $\partial U_3/\partial y$. Point 0 is on the boundary where $U_1 = U_3 = 0$ and equation (4.8) gives

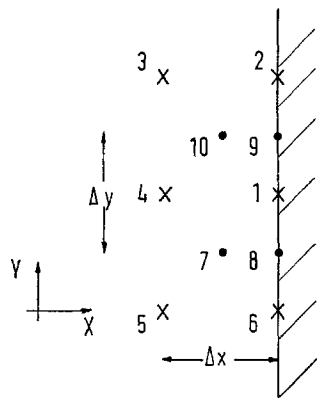
$$(U_2)_0 = (U_2)_1 - (U_1)_1 - \lambda(U_{3,y})_1 \tag{4.9}$$

where $\lambda = c\Delta t/2$.

Similarly at the other boundaries:

$$y = 0, \quad (U_3)_0 = (U_3)_4 + (U_1)_4 + \lambda(U_{2,x})_4, \tag{4.10}$$

$$y = 1, \quad (U_3)_0 = (U_3)_2 - (U_1)_2 - \lambda(U_{2,x})_2. \tag{4.11}$$



X ARE POINTS AT $t=t_n$

• ARE POINTS AT $t=t_{n+\frac{1}{2}}$

Figure 4. Conservation at a boundary.

The evaluation of quantities at points such as 1, 2, 3 or 4 in Figure 3 may be described by considering point 1 adjacent to a boundary through points 2, 5 and 4. Approximations to $\partial U_i/\partial y$ at internal nodes may be obtained, when required, by the use of central differences in the y direction. Approximations to U_i or $\partial U_i/\partial y$ at point 1 are obtained, as in the one dimensional case, using quadratic interpolation over point 5 and the two nodes to the left of 5 on the line through 1 and 5. Again, as in the one dimensional case, the stability condition imposed in the interior region will be an essential condition for stability of the boundary method.

D) *Conservation boundary method*

Integrating equation (3.5) over any area S in the $x-y$ plane we have

$$\frac{d}{dt}\int_S \boldsymbol{U}dS = -\int_S \boldsymbol{F}_{,x}dS - \int_S \boldsymbol{G}_{,y}dS. \tag{4.12}$$

For the boundary at $x = 1$ we use the mesh in Figure 4 where the crosses are at time t_n and the dots are at the intermediate time $t_{n+\frac{1}{2}}$. Take S to be the rectangle 1, 4, 5, 6 and let $\Delta\boldsymbol{U}$ denote an approximation to the change in $\boldsymbol{U}$ at any point in S during the time interval from t_n to $t_{n+\frac{1}{2}}$. Evaluating the integrals on the right hand side of (4.12) at time t_n, we have

$$\Delta\boldsymbol{U} = -\frac{\Delta t}{2\Delta x\Delta y}\left[\int_6^1 \boldsymbol{F}^n dy - \int_5^4 \boldsymbol{F}^n dy - \int_5^6 \boldsymbol{G}^n dx + \int_4^1 \boldsymbol{G}^n dx\right],$$

where $\int_6^1 \boldsymbol{F}^n dy$, for example, denotes the line integral of $\boldsymbol{F}$ along the straight line from point 6 to point 1 at time t_n. If the integrals are approximated using the trapezoidal rule this becomes

$$\Delta\boldsymbol{U} = -\frac{\Delta t}{4\Delta x}[\boldsymbol{F}_1^n + \boldsymbol{F}_6^n - \boldsymbol{F}_4^n - \boldsymbol{F}_5^n] - \frac{\Delta t}{4\Delta y}[\boldsymbol{G}_1^n + \boldsymbol{G}_4^n - \boldsymbol{G}_5^n - \boldsymbol{G}_6^n] \tag{4.14}$$

in the obvious notation. Using the average value of $\boldsymbol{U}$ over the four corners of S at t_n as an approximation to $\boldsymbol{U}_7^n$, we now obtain $\boldsymbol{U}_7^{n+\frac{1}{2}} = \frac{1}{4}(\boldsymbol{U}_1^n + \boldsymbol{U}_4^n + \boldsymbol{U}_5^n + \boldsymbol{U}_6^n) + \Delta\boldsymbol{U}$, which of course, is the value given by the first step of the L–W scheme (4.1) for the approximation at the intermediate point 7. If $\frac{1}{2}(\boldsymbol{U}_1^n + \boldsymbol{U}_6^n)$ is used as an approximation to $\boldsymbol{U}_8^n$ we may write

$$\boldsymbol{U}_8^{n+\frac{1}{2}} = \tfrac{1}{2}(\boldsymbol{U}_1^n + \boldsymbol{U}_6^n) + \Delta\boldsymbol{U},$$

and $\boldsymbol{U}_8^{n+\frac{1}{2}}$ may be expressed in terms of the computed value $\boldsymbol{U}_7^{n+\frac{1}{2}}$ as

$$\boldsymbol{U}_8^{n+\frac{1}{2}} = \boldsymbol{U}_7^{n+\frac{1}{2}} + \tfrac{1}{4}(\boldsymbol{U}_1^n + \boldsymbol{U}_6^n - \boldsymbol{U}_4^n - \boldsymbol{U}_5^n). \tag{4.15}$$

Similarly

$$\boldsymbol{U}_9^{n+\frac{1}{2}} = \boldsymbol{U}_{10}^{n+\frac{1}{2}} + \tfrac{1}{4}(\boldsymbol{U}_1^n + \boldsymbol{U}_2^n - \boldsymbol{U}_3^n - \boldsymbol{U}_4^n). \tag{4.16}$$

Approximations are now obtainable at points 7, 8, 9 and 10 at time $t_{n+\frac{1}{2}}$ and we may apply the conservation concept described above over the rectangle defined by these points between times t_n and t_{n+1}. Evaluating the fluxes $\boldsymbol{F}$ and $\boldsymbol{G}$ at time $t_{n+\frac{1}{2}}$, we find

$$\boldsymbol{U}_1^{n+1} = \boldsymbol{U}_1^n - \frac{\Delta t}{\Delta x}[\boldsymbol{F}_8^{n+\frac{1}{2}} + \boldsymbol{F}_9^{n+\frac{1}{2}} - \boldsymbol{F}_7^{n+\frac{1}{2}} - \boldsymbol{F}_{10}^{n+\frac{1}{2}}]$$

$$- \frac{\Delta t}{2\Delta y}[\boldsymbol{G}_9^{n+\frac{1}{2}} + \boldsymbol{G}_{10}^{n+\frac{1}{2}} - \boldsymbol{G}_7^{n+\frac{1}{2}} - \boldsymbol{G}_8^{n+\frac{1}{2}}]. \tag{4.17}$$

The other boundaries are treated in an analogous manner.

In applying the above method the external boundary conditions may be used at intermediate boundary points such as 8 or 9 and only the unknown components of $\boldsymbol{U}$ computed (I), or the complete vector $\boldsymbol{U}$ may be computed using (4.15) and (4.16) (II). Both (I) and (II) have been used in the comparative study which is under consideration here. It should be noted that the boundary methods derived in this section have only first order accuracy and are therefore less accurate than the method used in the interior.

E) *Semi-characteristic boundary method*

This method uses the theory of characteristics, but a finite difference scheme is used rather than an integration along the characteristics. If in equation (3.4)—repeated as equation (4.18) for convenience—

$$\frac{\partial \boldsymbol{U}}{\partial t} + A\frac{\partial \boldsymbol{U}}{\partial x} + B\frac{\partial \boldsymbol{U}}{\partial y} = \boldsymbol{0} \tag{4.18}$$

we regard the term $B\,\partial\boldsymbol{U}/\partial y$ as an inhomogeneous term in an equation involving x and t derivatives only then the system may be uncoupled as in the one dimensional case. Let P be the square matrix which we introduced in Section 2 having its ith row equal to a left eigenvector corresponding to the ith eigenvalue of A. The choice

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \text{ gives } P^{-1} = \tfrac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 2 & 0 \end{bmatrix} \text{ and}$$

$$PAP^{-1} = D = \begin{bmatrix} -c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{bmatrix}.$$

Pre-multiplying equation (4.18) by P and putting $V^T = (V_1, V_2, V_3) = PU$ we find

$$\frac{\partial V}{\partial t} + D\frac{\partial V}{\partial x} + PBP^{-1}\frac{\partial V}{\partial y} = \mathbf{0}, \tag{4.19}$$

where

$$PBP^{-1} = \begin{bmatrix} 0 & -c & 0 \\ -\frac{c}{2} & 0 & -\frac{c}{2} \\ 0 & -c & 0 \end{bmatrix}.$$

Now consider the boundary $x = 1$ where the conditions $U_1 = U_3 = 0$ imply that $V_2 = 0$ and $V_1 = -V_3$. In the format of equation (2.3) we may write the conditions at $x = 1$ as $[V_1] = [0\ -1][V_2, V_3]^T$. As suggested in Section 2, V_2 and V_3 are obtained by integrating along the non-negative characteristics and V_1 is then given by the equation above. The condition $V_2 = U_3 = 0$ has arisen, of course, from a trivial integration along the characteristic which lies on the boundary $x = 1$ and we must now use the outwardly directed positive characteristic to solve for V_3. At the boundary $x = 1$ we therefore use the third component of (4.19)

$$\frac{\partial V_3}{\partial t} + c\frac{\partial V_3}{\partial x} - c\frac{\partial V_2}{\partial y} = 0, \tag{4.20}$$

where $V_1 = U_1 + U_2$, $V_2 = U_3$ and $V_3 = U_1 - U_2$. We propose to evaluate the "inhomogeneous" term $c(\partial V_2/\partial y)$ at the boundary point at time $t_{n+\frac{1}{2}}$ (Figure 5) and since $V_2 \equiv 0$ on the boundary this term vanishes and (4.20) becomes

$$\frac{\partial V_3}{\partial t} + c\frac{\partial V_3}{\partial x} = 0. \tag{4.21}$$

In the extension of this method to the general nonlinear situation the eigenvalue c and the inhomogeneous term would be evaluated at the boundary point. In our constant coefficient linear problem equation (4.21) is similar to the one dimensional characteristics equation. Forward differences are used for the time derivative and a one sided second order difference scheme is used for the space derivative. The one sided difference scheme can be used at the boundary since the mesh points used are on the same side of the boundary as the characteristic associated with equation (4.21).

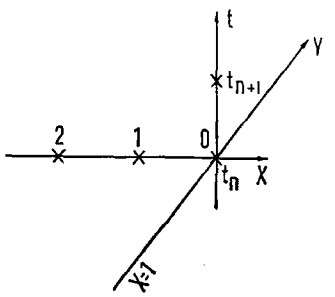


Figure 5. Semi-characteristic methods at a boundary.

A typical mesh at the boundary $x = 1$ is given in Figure 5. An initial estimate for $[V_3]_0^{n+1}$ is obtained using

$$[V_3]_0^{n+1} = [V_3]_0^n + \frac{c\Delta t}{2\Delta x} J^n \tag{4.22}$$

where

$$J^n = 4[V_3]_1^n - 3[V_3]_0^n - [V_3]_2^n. \tag{4.23}$$

A more accurate value is then obtained using

$$[V_3]_0^{n+1} = [V_3]_0^n + \frac{c\Delta t}{4\Delta x}[J^n + J^{n+1}] \tag{4.24}$$

where J^{n+1} is given by equation (4.23) with n replaced by $n + 1$. The internal mesh points must be calculated before the boundary points. For a nonlinear system the analogue of (4.24) would be used iteratively until successive iterates agreed to the required accuracy. Coefficients would be evaluated by averaging over the boundary points at t_n and t_{n+1}. For the constant coefficient problem under discussion, however, no iteration is required. A similar method is used for the boundaries at $y = 0,1$ on replacing the eigenvalues and eigenvectors of A by those for B. For the one dimensional case the equations are completely uncoupled and either the y dependence in equation (4.19) is ignored or an equation similar to (2.2) is used.

5. Discussion of results

The numerical solution to the $1D$ problem is computed using 21 mesh points in the interval $0 \leqq x \leqq 1$ and the results are tabulated in Table 1.

The numerical solution to the $2D$ problem is computed using 21×21 mesh points in the region $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$ for up to 100 time steps, 34 time cycles or 4.75 seconds and the results are tabulated in Table 2. The difference in computing time between the methods is negligible, 100 steps taking about 235 seconds of central processor time on an ICL 1904S.

The zero order extrapolation at the boundaries gives better results in $1D$ and $2D$ than was expected from this very crude method. Using this method for the $2D$ problem the re-

TABLE 1

Maximum absolute errors for the one dimensional problem

Boundary method	*Maximum absolute error*			
	40 *time steps*		100 *time steps*	
	Interior	*Boundary*	*Interior*	*Boundary*
A Analytic	$1.9 * 10^{-4}$	0	$2.5 * 10^{-4}$	0
B Zero extrapolate	$5.9 * 10^{-3}$	$6.0 * 10^{-3}$	$2.6 * 10^{-3}$	$2.6 * 10^{-3}$
C Characteristics	$6.7 * 10^{-4}$	$6.7 * 10^{-4}$	$2.0 * 10^{-3}$	$2.1 * 10^{-3}$
D Conservation I	$7.2 * 10^{-4}$	$7.2 * 10^{-4}$	$2.2 * 10^{-3}$	$2.2 * 10^{-3}$
II	$5.6 * 10^{-3}$	$4.9 * 10^{-3}$	$1.7 * 10^{-2}$	$1.2 * 10^{-2}$
E Semi-characteristics	$4.5 * 10^{-4}$	$4.3 * 10^{-4}$	$1.0 * 10^{-4}$	$9.7 * 10^{-4}$

TABLE 2

Maximum absolute errors for the two dimensional problem

Boundary method	*Maximum absolute error*			
	40 *steps* (1.3 *time cycles*)		100 *steps* (3.4 *time cycles*)	
	Interior	*Boundary*	*Interior*	*Boundary*
A Analytic	$3.1 * 10^{-2}$	0	$3.4 * 10^{-2}$	0
B Zero extrapolation	0.1	0.12	0.27	0.27
C Characteristics	$6.1 * 10^{-2}$	$5.7 * 10^{-2}$	0.14	0.14
D Conservation I	$6.6 * 10^{-2}$	$6.6 * 10^{-2}$	0.16	0.16
II	0.42	$2.1 * 10^{-2}$		
E Semi-characteristics	$6.2 * 10^{-2}$	$6.0 * 10^{-2}$	0.15	0.14

sults in Table 2 have $U_1 = 0$ on all the boundaries and $U_2 = 0$ on $y = 0,1$ and $U_3 = 0$ on $x = \pm 1$, but the results are somewhat better if $U_1 = 0$ is the only boundary condition.

Using conservation theory at the boundary of the $2D$ problem, if the intermediate values are taken from the boundary conditions then the solution is far more accurate than when the intermediate results are calculated. The same is also true for the $1D$ problem.

Results seem to indicate that the boundary method which should be used with a second order Lax–Wendroff interior method should be based on an approximation of the characteristics at the boundary. The characteristic method analogous to that described by Butler [9] seems to be one of the best boundary methods. The semi-characteristic method described in the paper seems to retain many of the advantages of a characteristics method and, of course, it is much easier to implement. One might expect that this method would be an ideal method to use in conjunction with the 2-step Leap Frog method, described in Richtmyer and Morton [3]. Employing three time levels, an explicit boundary method could be constructed having second order accuracy in space and time.

This paper discusses various boundary methods and tests them on a linear equation. Nonlinear equations and boundaries with re-entrant corners will give the boundary methods a more realistic test and the authors hope to do this in the future.

REFERENCES

[1] D. E. Koster, Stability of a finite difference scheme with "wrong" boundary conditions, *Siam. J. Num. Anal.* 10 (1973) 1039–1046.

[2] P. J. Roache, *Computational Fluid Dynamics*, Hermosa Publishers, Albuquerque (1972).

[3] R. D. Richtmyer and K. W. Morton, *Difference methods for Initial Value Problems*, 2nd Ed. Wiley (Interscience), New York 1967.

[4] B. Gustafsson, H. O. Kreiss and A. Sundstrom, Stability theory of difference approximations for mixed initial boundary value problems II, *Maths Comp.* 26 (1972) 649–686.

[5] A. Gourlay and J. LL. Morris, Comparison of multistep formulations, *J. Comp. Phys.* (1970) 229–242.

[6] C. K. Chu and A. Sereny, Boundary conditions in finite difference fluid dynamic codes, *J. Comp. Phys.* 15 (1974) 476–491.

[7] A. Sundstrom, Note on a paper by Chu and Sereny, *J. Comp. Phys.* 17 (1975) 450–454.

[8] J. C. Wilson, Stability of Richtmyer type difference schemes in any finite number of space variables and their comparison with multistep Strang schemes. *J. Inst. Maths. Applics.* 10 (1972) 238–257.

[9] D. S. Butler, The numerical solution of hyperbolic systems of partial differential equations in three independent variables. *Proc. Roy. Soc.* A255 (1960) 232–252.